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ABSTRACT

A model for the integrated moving averages process of order one, IMA (1, 1), having a seasonal (cyclic) component is presented. The model incorporates a parameter for possible change in level of the process after intervention, following methods developed by Box and Tiao (1965), and Glass, Willson, and Gottmann (1972). Least-squares estimates and associated significance tests for the parameters of the model (in particular, the intervention parameter) are derived. The results of a computer study and an example from real data are given with analysis and interpretation of parameter estimates. Results of the theoretical derivation are extended to other models [IMA (1, 1) with deterministic drift, multi-component models], and limitations of the model are noted. (Author)

Estimation of Intervention Effects in Seasonal Time-Series

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Abstract

The time-series experimental paradigm is extended to include processes which have seasonal variation. A basic linear model is examined which includes a seasonal component and intervention effect component. Transformation of observations puts the model into general linear form, which is amenable to solution by the method of least squares. Estimates for parameters of the model are derived, and confidence intervals formed around them.

A second method for dealing with the seasonal component is discussed. This is based upon seasonal adjustment prior to time-series analysis. A third method, that of differencing, is noted.

Applicability of these approaches in various parametric situations is checked by the use of simulations. Computer-generated time-series processes with different error variances and amplitudes were analyzed using the methods discussed. Results suggest that the first method works best in cases where error variance and amplitude are of the same order of magnitude. Seasonal adjustment seems better for situations when the amplitude is much larger than the error variance. Differencing was a poorer method in all cases.

Many stochastic processes observed in the behavioral sciences have seasonal or cyclic components. Such periodic variation must be included in any model which attempts to explain such processes parametrically. A special case is the time-series experiment with seasonal component.

Glass, Willson, and Gottman (1972) have given a general approach to design and analysis of interrupted time-series experiments. The problem of periodic variation was not considered, but its solution follows from the work presented there.

Consider observation of a time-series process which follows a periodic upward and downward movement over time distinct from the normal variation in each observation. After a number of observations, an intervention is made into the process, and additional observations are made. Analysis of the data is now required. Certainly the observer is interested in changes in the process due to intervention. He might also be interested in estimates of the original level, period and amplitude of the seasonal variation, and in any changes in the process after intervention. Mathematically, a discrete stochastic process may be modelled by an auto-regressive-integrated moving-averages process (ARIMA), following Box and Jenkins (1970). Glass et al (1972) note that for most processes, a simple model suffices: the integrated-moving averages process of order 1 or 2, IMA (1,1) or IMA (2,2). The IMA (1,1) is a stationary process (doesn't wander around much), in which random shocks or errors change the level of the process, and then die out, except for a portion of each previous shock which remains permanently. For IMA (2,2) (which is non-stationary and wanders from level to level), a portion of each two previous shocks remains. Only IMA (1,1) will be considered for the derivation of the model, but extension to IMA (2,2) is straightforward.

Method I

At each time t prior to intervention, the realization of an IMA (1,1) process with seasonal variation may be represented as a linear sum of terms, following Box and Jenkins (1972):

$$z_t = L + \lambda_0 \sum_{i=1}^{t-1} a_i + A \sin \omega_t + a_t \tag{1}$$

where L = level of process at the initial observation

λ_0 = proportion of each shock (error) a_i retained in the series

A = amplitude of the seasonal component

ω_t = argument of the sine term

a_i = random shock (error) at time i, and

$E(a_i) = 0$

$$\text{cov}\ (a_i a_j) = \sigma_a^2 I.$$

The sine term is just the first term of a Fourier representation of a seasonal variation. It must be noted that such a term is deterministic, and as such more useful in fitting data than in forecasting future data (see Box and Jenkins, 1970; p. 301). The choice of a sine term rather than cosine term is arbitrary and changes only the phase estimate.

After intervention an additional term devoting change is needed. In the simplest case this will be a change in level, δ. Then,

$$z_t = L + \lambda_0 \sum_{i=1}^{t-1} a_i + A \sin \omega_t + \delta + a_t \qquad (2)$$

The argument of the sine term will be as function of π, generally including a phase term:

$$\omega_t = \frac{2(t + K)}{P} \pi \qquad (3)$$

when t = time of observation

K = phase constant

P = period of the seasonal variation.

The model is linear in the sense of additivity of components, and is thus amenable to solution by the method of least-squares, after transformation. The additional error or shock terms, $\lambda_0 \sum_{i=1}^{t-1} a_i$, must be removed first. Box and Tiao (1965) first showed the form of the transformation, and Kepke (1972) has given a generalized method for any ARIMA process. For IMA (1,1), the transformed observation y_t is given by

$$\begin{aligned} y_1 &= z_1 \\ y_2 &= z_2 - z_1 - (\lambda_0 - 1)y_1 \\ &\vdots \\ y_t &= z_t - z_{t-1} - (\lambda_0 - 1)y_{t-1} \end{aligned} \qquad (4)$$

which may be expressed

$$y_t = (1 - \lambda_0)^{t-1}L + (1 - \lambda_0)^{t-1}A(\sin \omega_t) + a_t \qquad (5)$$

prior to intervention, and

$$y_t = (1 - \lambda_0)^{t-1}L + (1 - \lambda_0)^{t-1}Af(\sin \omega_t) + (1 - \lambda_0)^{t-n_1-1}\delta + a_t \quad (6)$$

after intervention. Notice that n_1 observations were made prior to intervention, and this fact is included in the intervention term. The y_t's are now in the form of the general linear model

$$\underline{Y} = X\underline{\beta} + E, \text{ where}$$

$$\begin{array}{r} \\ X = \\ \\ \\ \\ \text{Intervention....} \\ \\ \\ \\ \\ \\ \end{array} \left| \begin{array}{ccc} 1 & \sin \omega_1 & 0 \\ (1-\lambda) & f(\sin \omega_2) & 0 \\ (1-\lambda)^2 & f(\sin \omega_3) & 0 \\ \vdots & \vdots & \vdots \\ (1-\lambda_0)^{n_1-1} & f(\sin \omega_{n_1}) & 0 \\ \hdashline (1-\lambda_0)^{n_1} & & 1 \\ \vdots & \vdots & (1-\lambda_0) \\ \vdots & \vdots & (1-\lambda_0)^2 \\ \vdots & & \vdots \\ (1-\lambda_0)^{n_1+n_2-1} & f(\sin \omega_{n_1+n_2}) & (1-\lambda_0)^{n_2-1} \end{array} \right| \quad (7)$$

$$\beta = \begin{bmatrix} L \\ A \\ \delta \end{bmatrix} \quad (8)$$

and

$$f(\sin \omega_t) = \lambda_0 \sum_{i=0}^{t-2}(1 - \lambda_0)^i \sin \omega_{t-i-1} + \sin \omega_t$$

Note that the amplitude is being estimated and the period is assumed to be known. Practically, this will nearly always be true.

The least-squares estimates $\hat{\beta}$ are given by

$$\hat{\beta} = (X'X)^{-1}X'y. \tag{9}$$

No distribution theory for the errors has been mentioned. The assumption of normality, generally reasonable, allows calculation of confidence intervals about the estimates $\hat{L}$, $\hat{A}$, and $\hat{\delta}$. The standard errors are given by

$$s_{\hat{\beta}_j} = s_a\sqrt{c^{jj}}$$

where c^{jj} is the jth diagonal entry of $(X'X)^{-1}$, and $s_a^2 = \frac{\hat{a}'\hat{a}}{n_1+n_2-2}$, the estimated error variance of the a_i. The $100(1 - \alpha)\%$ confidence interval is

$$\hat{\beta}_i \pm {}_{1-\alpha/2}t_{n_1+n_2-2}s_e\sqrt{c^{jj}} \tag{10}$$

The derivation was made on the assumption of known parameter λ_o. In general this will not be met, and an iterative procedure must be used to estimate λ_o. Solutions of (9) may be made for values of λ_o (0)(.01)(2.0) economically by computer, and a minimum variance criterion used to find the best solution, with an estimate $\hat{\lambda}_o$ of λ_o. Box and Tiao (1965) also show a Bayesian solution in which the posterior density function is maximized for $\hat{\lambda}_o$ (see Glass et al 1972).

The model for IMA (2,2) is slightly more complicated. Here, the process may wander from level to level, and it is thought that on theoretical grounds solution will be much less satisfactory since a seasonal term may well be confounded with random changes in level. It is mentioned for completeness. The

z_t are given by

$$z_t = L + \lambda_1 \sum_{i=1}^{t} a_i + \lambda_2 \sum_{i=1}^{t-1} \sum_{j=1}^{i} a_j + A \sin \omega_t + a_t \qquad (11)$$

Intervention will produce a δ term.

The transformation is slightly more complicated and was derived by Glass (1972). The transformed observation y_t are given by

$$y_1 = z_1$$

$$y_2 = z_2 - 2z_1 - (\lambda_1 + \lambda_2 - 2)y_1$$

$$\vdots$$

$$y_t = z_t - 2z_{t-1} - (\lambda_1 + \lambda_2 - 2)y_{t-1} - (1 - \lambda_1)y_{t-2} \qquad (12)$$

and again, least-squares estimates may be calculated. Iterations over both λ_1 and λ_2 produce a two-dimensional plane of variances, and contour lines of equal variance may be used to estimate λ_1 and λ_2 (see Glass, 1972 , and Box and Jenkins, 1970, p. 212-213.)

Method II

Another approach to the problem of seasonality is to estimate the amplitude and period prior to least-squares analysis and remove the seasonal term, leaving a residual which may be analyzed as an IMA (1,1) process. Thus

$$z_t' = z_t - \hat{A} \sin \hat{\omega}_t \qquad (13)$$

The estimate $\hat{A}$ may be made by

$$\hat{A} = \frac{\max(z_t) - \min(z_t)}{2} \text{ for } (\nu_1 \leq t \leq \nu_2)$$

where t is in some restricted range. $\hat{A}$ can be shown to be unbiased. The new observations z'_t are now treated as a simple IMA (1,1) process (or IMA (2,2) if that is the case), and the $z'_t \rightarrow y'_t$ by the transformation given in (4). Least-squares estimates for L and δ may then be computed.

Method III

Box and Jenkins (1970) suggest the method of differencing to cope with seasonality. This in effect ignores the seasonal term by looking at residuals.

$$z'_t = z_{t+p} - z_t \tag{14}$$

where p = period of the seasonal component.

The residuals are treated as an IMA (1,1) process. The original intent of this method lay in getting better estimates in forecasting, and a more complicated multiplicative model was built from the pth differences. Such a model is not comtemplated here, and the differencing was considered the final step in this method.

Method IV

Yet another method of analysis is to ignore the seasonal component entirely and analyze the process as if it were a non-seasonal IMA (1,1). The seasonal component will increase the error variance, and give wider confidence intervals.

Comparison of Methods

The four methods were used to analyze seven simulated processes which were identical except that error variance and amplitude were varied. Table 1 gives a summary of the analyses. The processes were all IMA (1,1) processes built according to equation (1). The seven processes are shown in Figure 1.

Inspection of results for Case I reveals method I to have estimated the process parameter (λ_0) and error variance best, although all estimates of L and δ are good. The results for Case 2 again show method I to be best, with Methods II and IV somewhat worse. In Case 3 method II shows an error variance closer to the actual case and better estimate of δ, although method I gives a better estimate of amplitude. In Case 4 no method seems clearly superior to another, and all do rather poorly. Method I shows better estimates of A and λ_0 than the others. Case 5 clearly demonstrates the superiority of analysis by Method II in the situation of small error variance and large sine term, and this is also true in Case 6 and Case 7, although better estimates by method I occur in these cases than in Case 5.

The results suggest this conclusion: Method I is better when a small amplitude seasonal component exists with respect to error variance (same order of magnitude); Method II is better when a large seasonal component is present (with respect to error variance).

It is apparent that analysis is not nearly so precise as is possible without the existence of seasonal terms. The periodic component tends to confound examination of error variance, and quite probably one must be content with results less exact than is possible in other cases.

Although no example was found which adequately fit the assumptions of the IMA (1,1) model, an analysis is presented to illustrate the procedure. Airline

Table 1

Comparison of Estimates by Four Methods for Seven Simulated Seasonal Processes

	Parameters	Method I (Linear sine term included) Estimates	Method II (Seasonally Adjusted) Estimates	Method III (Differenced) Estimates	Method IV (IMA (1,1) Undifferenced) Estimates
Case 1	L=50	49.6	48.7	50.5	~51
	δ=20	19.2	19.2	16.2	~19
	λ=.7	.65	.88 (p=2)	~0	0
	σ^2=1	1.3	1.9	1.1	~1
	A=1	1.0	2.5		
	t-statistic for δ=0	17.7*	13.75*		
Case 2	L=50	53.1	48.7	52.3	54.1
	δ=20	22.1	21.6	~10	19.7
	λ=.7	.7	1.01 (p=2)	~0	.55
	σ^2=25	34.0	46.3	~60	31.3
	A=1	2.6	6.5		
	t-statistic for δ=0	3.60*	2.58*		
Case 3	L=50	44.5	42.3	43.9	43.9
	δ=20	31.2	20.1	25.5	25.5
	λ=.7	.95	.54	.95	.95
	σ^2=25	46.1	36.2	51.2	51.2
	A=10	10.2	8.5		
	t-statistic for δ=0	4.52*	3.71*		
Case 4	L=50	42.0	36.2	54.9	45.7
	δ=20	31.2	11.3	14.4	17.3
	σ^2=25	48.1	39.8	51.2	83.1
	λ=.7	.90	1.03	1.21	1.40
	A=20	18.3	25.8		
	t-statistic for δ=0	6.94*	1.76		
Case 5	L=50	37.2	43.2	41	35.5
	δ=20	18.6	18.3	18	15.4
	λ=.7	1.65	.51	0	1.81
	σ^2=1	19.0	3.4	5	27.6
	A=20	7.9	20.0		
	t-statistic for δ=0	5.62*	10.65*		
Case 6	L=50	50.0	43.1	54.1	56.1
	δ=20	36.4	22.5	27.0	34.0
	λ=.7	1.1	.88	.79	1.45
	σ^2=16	29.7	18.5	26.7	49.9
	A=20	13.4	24.0		
	t-statistic for δ=0	6.50*	5.14*		

Table 1

Comparison of Estimates by Four Methods for Seven Simulated Seasonal Processes

	Parameters	Method I (Linear sine term included) Estimates	Method II (Seasonally Adjusted) Estimates	Method III (Differenced) Estimates	Method IV (IMA (1,1) Undifferenced) Estimates
Case 7	L=50	52.0	51.2	60	53.0
	δ=20	27.6	20.6	30	27.0
	λ=.7	.8	.61	0	.47
	σ^2=25	24.6	22.5	21	19.7
	A=5	5.2	5.5		
	t-statistic for δ=0	5.55*	4.50*		

$_{.995}t_{40} = 2.70$

*$p < .01$

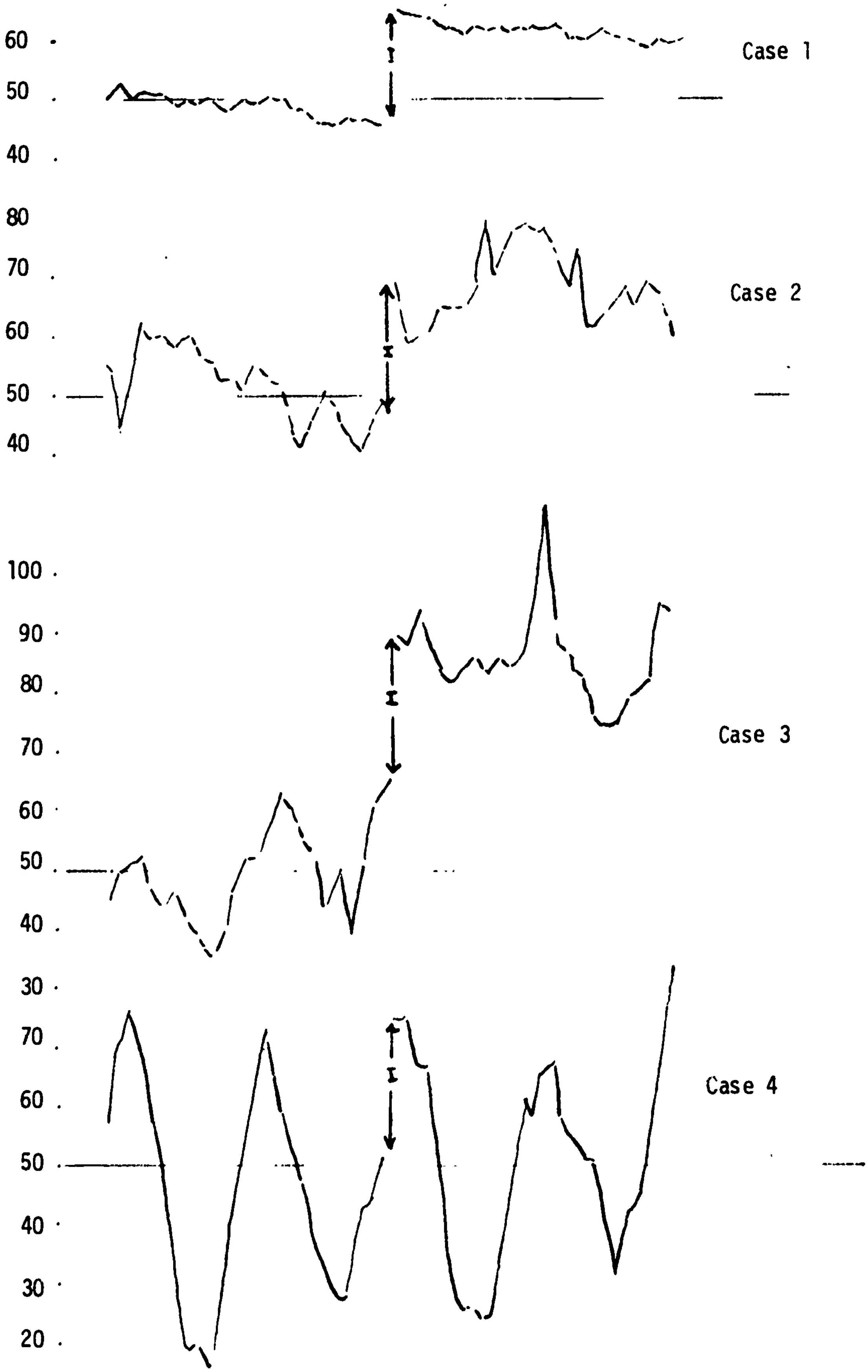

Figure 1. Seven Simulated Seasonal Time-Series Processes with Intervention After 25 Observations.

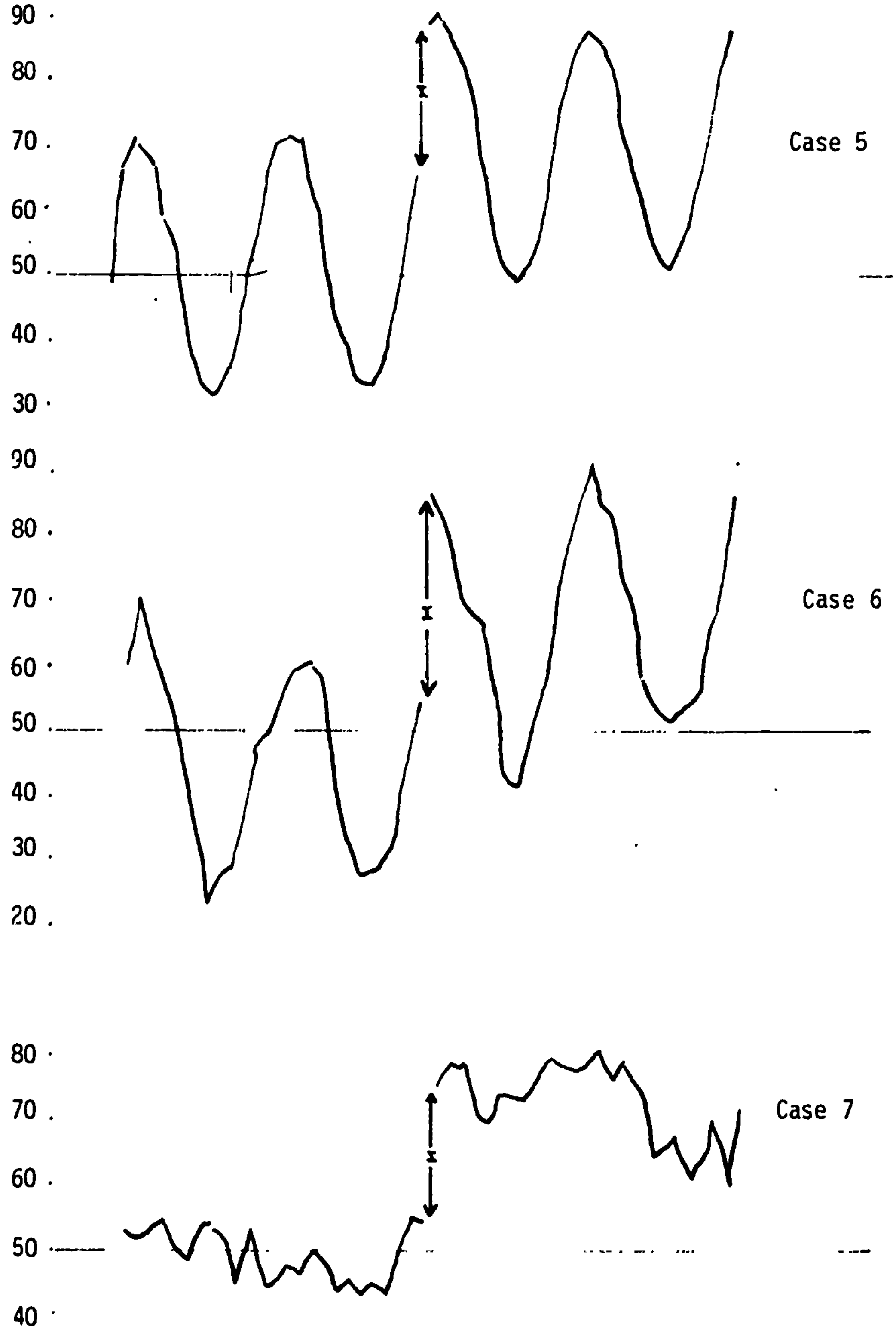

Figure 1. (continued) Seven Simulated Seasonal Time-Series Processes with Intervention after 25 Observations.

passengers carried by major airlines between 1949 and 1951 by month vary seasonally. No real intervention is thought to have occurred, but intervention is hypothesized after January 1950. Thus, $\delta = 0$ for the analysis. The graph of the data (from Box and Jenkins, 1970), is given in Figure 2. Parameter estimates for methods I and II are summarized in Table 2.

The error variances are comparable in both methods, but the amplitude is better estimated in method II. Inspection of the graph suggests the process to have amplitude larger than error variance, thus implying better fit will be gained from method II. The intervention effect was non-significantly different from zero in both cases.

Analysis of the data by Box and Jenkins (1970) was performed using a much more complicated multiplicative difference model. Error variances are not reported.

Summary

The use of simple trigonometric terms to account for seasonality may have utility in analysis of time-series experimental data. Additional complications of the linear model by complex components may well obscure affects of intervention. Two methods of analysis are recommended. The first, analysis with a sine (or cosine) term included in least-square estimation works well with moderate seasonality -- that is, error variance comparable to the amplitude of the sine component. The second method, seasonal adjustment prior to analysis, works well with seasonal variation large with respect to error variance.

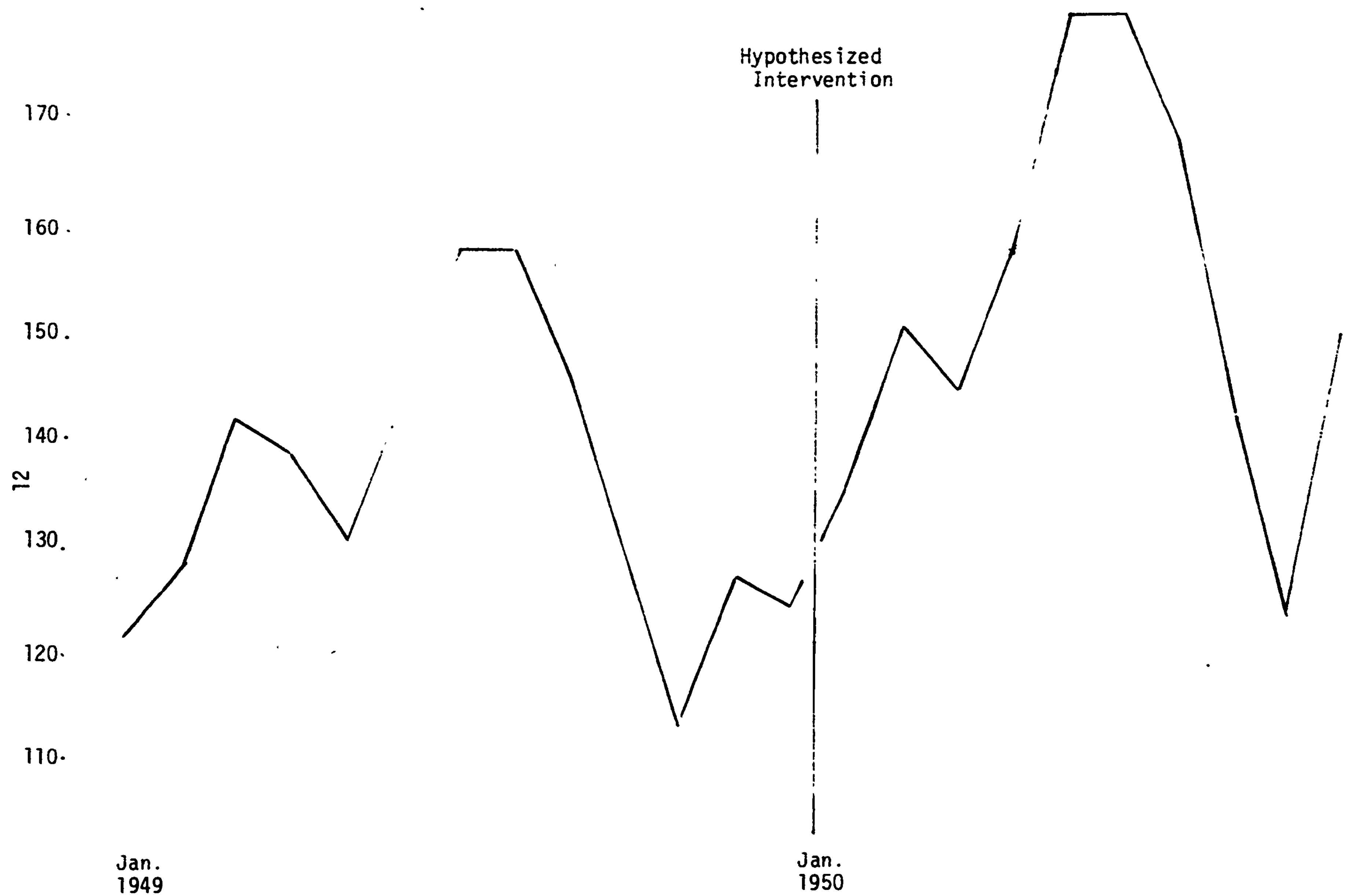

Figure 2. Total Number of Passengers Carried (in thousands) 1949-1951 (from Box and Jenkins, 1970).

Table 2

Parameter Estimates for Total Airline Passengers
Carried Between 1949 and 1951

	Method I	Method II
$\hat{L}$	111.3	134.2
$\hat{\delta}$	9.2*	5.4*
$\hat{\sigma}_a^2$	174.0	178
$\hat{A}$	7.9	24.0
$\hat{\lambda}_o$	1.45	1.4

* $p > .1$ for $\delta = 0$

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